SOLUTIONS TO EXERCISES 6.3.11, 6.4.11, AND 6.5.3 FROM PROBLEM SET 7

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1. Ex. 6.3.11: *Find the closest point to* $\mathbf{x}$ *in the subspace* W *spanned by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Solution. Because $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 - 1 - 1 - 1 = 0$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set. Beause neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ is $\mathbf{0}$, this means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Since W is spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$, we have that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

Therefore, the answer is $\operatorname{Proj}_W \mathbf{x}$, and we can calculate it as:

$$\frac{\mathbf{x}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}\cdot\mathbf{v}_2}{\mathbf{v}_1\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{9+1-5+1}{9+1+1+1}\mathbf{v}_1 + \frac{3-1+5-1}{1+1+1+1}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}}.$$

□

2. Ex. 6.4.11: *Find an orthogonal basis for the column space of the following matrix:*

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution. Let A be the matrix in the problem, let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be its three columns, and let V be $\operatorname{Col} A$. Then we want to find an orthogonal basis for V.

In order to use Gram-Schmidt as it's written in the book, we need a basis for V. By definition, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a spanning set for V, but we need to know that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent before we know it's a basis. To check that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent, we row-reduce A:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{Add }(-1)(\text{Row 1})\text{ to Rows 4 and 5}]{\text{Add Row 1 to Rows 2 and 3}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \\ 0 & -6 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[\text{Add 2(Row 2) to Row 4}]{\text{Add }(-2)(\text{Row 2})\text{ to Row 3}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{bmatrix}.$$

This means A has a pivot in every column, so the columns of A are linearly independent. In other words, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent.

Now we can use Gram-Schmidt on $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Let:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, -1, 1, 1).$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \frac{2-1-4-4+2}{1+1+1+1+1}\mathbf{v}_1 = \mathbf{x}_2 + \mathbf{v}_1 = (3, 0, 3, -3, 3).$$

$$\mathbf{v}_2 = \mathbf{x}_3 - \frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{5+4+3+7+1}{1+1+1+1+1}\mathbf{v}_1 - \frac{15+0-9-21+3}{9+0+9+9+9}\mathbf{v}_2 = \mathbf{x}_3 - 4\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2$$
$$= (5, -4, -3, 7, 1) + (-4, 4, 4, -4, -4) + (1, 0, 1, -1, 1) = (2, 0, 2, 2, -2).$$

Then:

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}}.$$

is an orthogonal basis of the column space of the matrix.

There are many other bases for the column space, including $\left\{\mathbf{v}_1, \frac{1}{3}\mathbf{v}_2, \frac{1}{2}\mathbf{v}_3\right\}$.

Also, although Lay's text doesn't say this, it is possible to use Gram-Schmidt on a list of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ that may not be linearly independent to find an orthogonal basis for $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. The only difference is that some of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ might be $\mathbf{0}$. However, if $\mathbf{v}_i = \mathbf{0}$, then if you drop the $-\frac{\mathbf{x}_j \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\mathbf{v}_i$ term when you calculate $\mathbf{v}_j$ for each $j > i$, then the nonzero vectors in $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ will be an orthogonal basis for $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. ☐

3. Ex. 6.5.3*: Find a least-squares solution of* $A\mathbf{x} = \mathbf{b}$ *by* (a) *constructing the normal equations for* $\hat{\mathbf{x}}$ *and* (b) *solving for* $\hat{\mathbf{x}}$.

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

Solution. (b) Compute:

$$A^T A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

The normal equations for $\hat{\mathbf{x}}$ is $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$, so the normal equation is $\boxed{\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}}$.

(b) Solve the normal equation:

$$\begin{bmatrix} 6 & 6 & 6 \\ 6 & 42 & -6 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } (-1)(\text{Row 1})} \begin{bmatrix} 6 & 6 & 6 \\ 0 & 36 & -12 \end{bmatrix} \xrightarrow[\text{Row 2 by } \frac{1}{36}]{\text{Multiply}} \begin{bmatrix} 6 & 6 & 6 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

$$\xrightarrow[\text{to Row 1}]{\text{Add } (-6)(\text{Row 2})} \begin{bmatrix} 6 & 0 & 8 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix} \xrightarrow[\text{Row 1 by } \frac{1}{6}]{\text{Multiply}} \begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

The only solution to the normal equation, which is also the only least-squares solution to $A\mathbf{x} = \mathbf{b}$, is $\hat{\mathbf{x}} = \boxed{\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}}$. ☐